Fractional Nonholonomic Ricci Flows

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Abstract

We formulate the fractional Ricci flow theory for (pseudo) Riemannian geometries enabled with nonholonomic distributions defining fractional integro–differential structures, for non–integer dimensions. There are constructed fractional analogs of Perelman's functionals and derived the corresponding fractional evolution (Hamilton's) equations. We apply in fractional calculus the nonlinear connection formalism originally elaborated in Finsler geometry and generalizations and recently applied to classical and quantum gravity theories. There are also analyzed the fractional operators for the entropy and fundamental thermodynamic values.

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1 Introduction

The purpose of this paper is to generalize the Ricci flow theory [1, 2, 3, 4] (see [5, 6, 7] for reviews of results and methods) to fractional evolution of geometries of non–integer dimensions. The most important achievement of this theory was the proof of W. Thurston's Geometrization Conjecture by Grisha Perelman [2, 3, 4]. The main results on Ricci flow evolution were proved, in the bulk, for (pseudo) Riemannian and Kähler geometries. We show that similar results follow for geometries with non–integer dimensions when a fractional differential and integral calculus is corresponding encoded into nonholonomic frame structures and adapted geometric objects.

1.1 Basic concepts and ideas

In a series of works (see, for instance, [8, 9, 10] and references therein), we proved that nonholonomic constraints on Ricci flow evolution may transform (pseudo) Riemannian metrics and Levi–Civita connections into (in general)

nonsymmetric metrics and (for instance) Lagrange–Finsler type linear connections[1]. Such geometries can be modeled by nonholonomic distributions and frames with associated nonlinear connection (N–connection) structures on (pseudo) Riemannian spaces and/or generalizations.[2]

If nonholonomic distributions on a manifold M contain corresponding integro–differential relations, we can model fractional geometries (for spaces with derivatives and integrals of non–integer order). The first example of derivative of order $\alpha = 1/2$ has been described by Leibnitz in 1695, see historical remarks in [16]. The theory of fractional calculus with derivatives and integrals of non–integer order goes back to Leibniz, Louville, Grunwald, Letnikov and Riemann [17, 18, 19, 20]. Derivatives and integrals of fractional order, and fractional integro–differential equations, have found many applications in physics (for example, see monographs [21, 22] and papers [23, 24, 25, 26]).

We consider that the question if analogous of Thurston (in particular, Poincaré) Conjecture can be formulated (and may be proven ?) for some spaces with fractional dimension is of fundamental importance in modern mathematics and physics. As a first step, the goal of this paper is to formulate a fractional version of the Hamilton–Perelman theory of Ricci flows. On Perelman's functionals, we shall follow the methods elaborated in Sections 1-5 of Ref. [2] but generalized for fractional nonholonomic manifolds by developing certain constructions from our works on nonholonomic Ricci flow evolution [8, 9].

We define a fractional nonholonomic manifold (equivalently, space) $\overset{\alpha}{\mathbf{V}}$ to be given by a quadruple $(\mathbf{V}, \overset{\alpha}{\mathcal{N}}, \overset{\alpha}{\mathbf{d}}, \overset{\alpha}{\mathbf{I}})$, with a fractional real number α, where $\mathbf{V}$ is a "prime" manifold of integer dimension and necessary smooth class and $\overset{\alpha}{\mathcal{N}}$ is a nonintegrable distribution correspondingly adapted with a fractional differential calculus $\overset{\alpha}{\mathbf{d}}$ and fractional calculus $\overset{\alpha}{\mathbf{I}}$ on $\mathbf{V}$. For such fractional nonholonomic geometries (including as particular cases, for instance, (pseudo[3]) Riemannian and Finsler geometries), we shall develop the approach to encode a corresponding fractional integro–differential calculus adapted to nonholonomic distributions. Perhaps, the simplest way is to fol-

[1] see reviews [11, 12] and monograph [13], and references therein, on modern developments and applications in modern physics of the geometry of nonholonomic manifolds and Lagrange–Finsler spaces [14, 15]

[2] There are used also equivalent terms like anholonomic and non–integrable manifolds. A nonholonomic manifold is defined by a pair $(\mathbf{V}, \mathcal{N})$, where $\mathbf{V}$ is a manifold and $\mathcal{N}$ is a nonintegrable distribution on $\mathbf{V}$.

[3] mathematicians use the term "semi"

low locally a fractional vector calculus with "combined" Riemann–Liouville and Caputo derivatives as in [24] resulting in a self–consistent fractional generalization of integral operations (and corresponding fractional Gauss's, Stokes', Green's etc integral theorems which, in our approach, are crucially important for constructing fractional Perelman's functionals).[4]

The article is organized as follows: In section 2, we provide a brief introduction into the geometry of fractional nonholonomic manifolds. Grisha Perelman's functional approach to Ricci flow theory is generalized for fractional nonholonomic manifolds in section 3. We derive the fractional nonholonomic evolution equations in section 4. A statistical interpretation of fractional nonholonomic spaces and Ricci flows is proposed. Formulas for fractional differential and integral calculus are summarized in Appendix.

1.2 Remarks on notations and proofs

1. We shall elaborate for fractional nonholonomic spaces a system of notations unifying that for the nonholonomic manifolds and bundles [12, 8, 9] and fractional integro–differential calculus [17, 18, 19, 20, 24] (we consider the reader to be familiar with the results of such works). For geometric objects spaces with nonholonomic distributions, we shall use boldface symbols like $\mathbf{V}, \mathbf{N}$ etc and put an up label α for fractional generalizations, for instance, of operators $\overset{\alpha}{\mathbf{d}}, \overset{\alpha}{\mathbf{I}}$. We do not use the symbol D for fractional partial derivatives as usually in works on fractional calculus (but we shall write $\overset{\alpha}{\partial}$, or $\overset{\alpha}{d}$) because on curved spaces of integer dimension the symbols D and $\mathbf{D}$ are used for covariant derivatives. It is convenient to keep the same notations of covariant derivatives for fractional curved spaces rewriting them, respectively, as $\overset{\alpha}{D}$, or $\overset{\alpha}{\mathbf{D}}$. We shall put a "fractional" label α on the left, like ${}^{\alpha}A$, if that will result in a more compact system of notations.

2. There will be also used "up" and "low" labels for some canonical geometric objects/operators, for instance, for the horizontal (h) and vertical (v) of a distinguished linear connection $\mathbf{D} = ({}_{h}D, {}_{v}D)$ and/or its fractional generalization $\overset{\alpha}{\mathbf{D}} = ({}_{h}\overset{\alpha}{D}, {}_{v}\overset{\alpha}{D})$. Splitting of space dimension

[4]There were formulated various approaches to fractional differential and integral calculus; on nonholonomic manifolds, some of them can be related via nonholonomic transforms/deformations of geometric structures. In this paper, we elaborate a formalism with Caputo fractional derivative (which gives zero acting on constants) adapted to nonlinear connections preserving a number of similarities with an unified covariant calculus for (pseudo) Riemannian manifolds and Lagrange–Finsler geometries [11, 12, 13].

$\dim \mathbf{V} = n+m$ is considered for a nonholonomic distribution (defining a nonlinear connection, N–connection, structure) $\mathbf{N} : T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}$, where $\oplus$ is the Whitney sum, $\dim(h\mathbf{V}) = 2n$ and $\dim(v\mathbf{V}) = 2m$. In some important particular cases, we can consider $\mathbf{V}$ to be a (pseudo) Riemannian manifold enabled with a nonholonomic distribution $\mathcal{N}$ induced by $\mathbf{N}$, or $V = E/TM$ for a vector/tangent bundle (E, π, M) / (TM, π, M) on a manifold M, $\dim M = n$, $\dim E = n+m$ / $\dim TM = 2n$, with π being a corresponding surjective projection. Indices of local coordinates on a point $\mathbf{u}$ on a open chart $\mathbf{U}$ for an atlas $\{U\}$ covering $\mathbf{V}$ are split in the form $\mathbf{u}^\alpha = (x^i, y^a)$, (or in brief $u = (x, y)$), where the general Greek indices $\alpha, \beta, \gamma, ...$ split correspondingly into h–indices $i, j, k, ... = 1, 2, ..., n$ and v–indices $a, b, c, ... = n+1, n+2, ..., m$. There are possible various types of transforms of local frames and coframes, $\mathbf{e}_\beta = (e_j, e_b)$ and $\mathbf{e}^\beta = (e^j, e^b)$ (in particular, coordinates with "primed", "underlined" indices etc), when, for instance, $\mathbf{e}_{\beta'} = e^{\beta}{}_{\beta'}(u)\mathbf{e}_\beta$ or $\mathbf{e}_\beta = e^{\underline{\beta}}{}_{\beta}(u)\mathbf{e}_{\underline{\beta}}$, $\mathbf{e}_{\underline{\beta}} = \partial_{\underline{\beta}} = (\partial_{\underline{j}} = \partial/\partial x^{\underline{j}}, \partial_{\underline{a}} = \partial/\partial y^{\underline{a}})$ for the Einstein's summation rule on indices being accepted. Geometric objects on $\mathbf{V}$, for instance, tensors, connections etc can be adapted to a N–connection structure and defined by symbols with coefficients on the right, running corresponding values with respect to N–adapted bases (preserving a chosen h–v–decomposition), for instance, $\mathbf{R} = \{\mathbf{R}^{\beta}{}_{\gamma\tau\mu} = \{R^{j}{}_{klm}, R^{b}{}_{klm}, R^{b}{}_{klc}, ...\}\}$.

3. Generalizing correspondingly a fractional integro–differential calculus from [24] to nonholonomic manifolds, we can elaborate a formal (abstract) analogy with the "integer" case but with certain modifications of the rules of local differentiation and "mixed" nonholonomic integral–differential rules. The "fractional" spaces and geometric objects will be enabled with an up label α in the form $\overset{\alpha}{\mathbf{V}}, \overset{\alpha}{\mathbf{D}}, \overset{\alpha}{\mathbf{e}}{}^{\beta}, \overset{\alpha}{\mathbf{R}}{}^{\beta}{}_{\gamma\tau\mu}$ etc.

4. Following the abstract fractional N–adapted calculus, the proofs of theorems became very similar to those given in N–adapted form [12, 8, 9], which was used for a nonholonomic generalization of the Ricci flow theory [1, 2, 5, 6, 7]. In this paper, we sketch proofs using the above mentioned formal analogy between N–adapted fractional and integer geometric constructions. Proofs of the fractional integro–differential theorems related to Ricci flow evolution became very sophisticate if we do apply the formalism of nonholonomic distributions, do not introduce nonlinear connections and do not apply certain methods from the geometry of nonholonomic manifolds.

2 Nonholonomic Manifolds with Fractional Distributions

The fractional differential calculus for flat spaces elaborated in [24, 25], and outlined in Appendix A, is extended for nonholonomic manifolds. For simplicity, such fractional manifolds can be modelled as real (pseudo) Riemannian spaces enabled with nonholonomic distributions containing such integro–differential relations when the fractional calculus on curved spaces with nonintegrable constraints is elaborated in a form maximally similar to integer dimensions.

2.1 Fractional (co) tangent bundles

For the integer differential calculus, the tangent bundle TM over a manifold can be constructed for a given local differential structure with standard partial derivatives ∂_i. Such an approach can be generalized to a fractional case when instead of ∂_i the differential structure is substituted, for instance, by the left Caputo derivatives ${}_{1x^i}\overset{\alpha}{\underline{\partial}}_i$ of type (A.1) for every local coordinate x^i on a local cart X on M.

Let us review, in brief, the definition for fractional tangent bundle $\overset{\alpha}{\underline{T}}M$ for $\alpha \in (0,1)$ (the symbol T is underlined in order to emphasize that we shall associate the approach to a fractional Caputo derivative). Here we cite the paper [28] for some similar constructions with fractional tangent spaces but for the left fractional RL derivative. We do not follow that approach because it is not suitable for elaborating fractional Ricci flow and gravitational models with exactly integrable evolution and, respectively, field equations, and a self–consistent fractional integral calculus with "simplified" integral theorems. For our purposes, it is more convenient to use the fractional calculus formalism proposed in Ref. [24].

We have a fractional Caputo left contact α in a point ${}_0x \in X$ for the parametrized curves on M parametrized by a real parameter τ and ${}_1c,\ {}_2c : I \to M$, with $0 \in I$; ${}_1c(0) = {}_2c(0) \in M$ if ${}_{1x^i}\overset{\alpha}{\underline{\partial}}_i(f \circ {}_1c)_{|\tau=0} = {}_{1x^i}\overset{\alpha}{\underline{\partial}}_i(f \circ {}_2c)_{|\tau=0}$ holds for all analytic functions f on X. This defines a relation of equivalence when the classes $[\overset{\alpha}{\underline{c}}]_{\ {}_0x}$ determines the fractional left tangent Caputo space $\overset{\alpha}{\underline{T}}_{\ {}_0x}M$. The corresponding fractional tangent bundle is $\overset{\alpha}{\underline{T}}M := \bigcup_{{}_0x} \overset{\alpha}{\underline{T}}_{\ {}_0x}M$ when the surjective projection $\overset{\alpha}{\pi} : \overset{\alpha}{\underline{T}}M \to M$ acts as $\overset{\alpha}{\pi}[\overset{\alpha}{\underline{c}}]_{\ {}_0x} = {}_0x$. Such a fractional bundle space is given by a triple $\left(\overset{\alpha}{\underline{T}}M, \overset{\alpha}{\pi}, M\right)$. For

simplicity, we shall write, in brief, for the total space only the symbol $\underline{\overset{\alpha}{T}}M$ if that will not result in ambiguities.

Locally on M, the class $[\underline{\overset{\alpha}{c}}]_{\ 0x}$ is characterized by a curve

$$x^i(\tau) = x^i(0) + \frac{\tau^\alpha}{\Gamma(1+\alpha)} \ {}_{1c}\underline{\overset{\alpha}{\partial}}_\tau x^i_{|\tau=0},$$

for $\tau \in (-\varepsilon, \varepsilon)$. So, the horizontal and vertical coordinates (respectively, h– and v–coordinates) on $\overset{\alpha}{\pi}{}^{-1}(X) \subset \underline{\overset{\alpha}{T}}_{\ 0x}M$ are $\overset{\alpha}{u}{}^{\beta} = (x^j, \overset{\alpha}{y}{}^{j})$, where

$$x^i = x^i(0) \text{ and } \overset{\alpha}{y}{}^{j} = \frac{1}{\Gamma(1+\alpha)} \ {}_{1c}\underline{\overset{\alpha}{\partial}}_\tau x^i(\tau)_{|\tau=0}.$$

For simplicity, we shall write instead of $\overset{\alpha}{u}{}^{\beta} = (x^j, \overset{\alpha}{y}{}^{j})$, the local coordinates $u^\beta = (x^j, y^j)$ both for integer and fractional tangent bundles considering that there were chosen certain such parametrizations of local coordinate systems by using classes of equivalence only for the left Caputo fractional derivatives.

On $\underline{\overset{\alpha}{T}}M$, we can consider an arbitrary fractional left (Caputo) frame basis

$$\underline{\overset{\alpha}{e}}_\beta = e^{\beta'}_{\ \beta}(u^\beta)\underline{\overset{\alpha}{\partial}}_{\beta'} \tag{1}$$

where the fractional local coordinate basis

$$\underline{\overset{\alpha}{\partial}}_{\beta'} = \left(\underline{\overset{\alpha}{\partial}}_{j'} = {}_{1x'} \underline{\overset{\alpha}{\partial}}_j, \underline{\overset{\alpha}{\partial}}_{b'} = {}_{1y^{b'}} \underline{\overset{\alpha}{\partial}}_{b'}\right) \tag{2}$$

is with running of indices of type $j' = 1, 2, ..., n$ and $b' = n+1, n+2, ..., n+n$. We might introduce arbitrary fractional co–bases which are dual to (1),

$$\underline{\overset{\alpha}{e}}{}^{\beta} = e_{\beta'}^{\ \beta}(u^\beta)\overset{\alpha}{d}u^{\beta'}, \tag{3}$$

where the fractional local coordinate co–basis

$$\overset{\alpha}{d}u^{\beta'} = \left((dx^{i'})^\alpha, (dy^{a'})^\alpha\right) \tag{4}$$

with h– and v–components, $(dx^{i'})^\alpha$ and $(dy^{a'})^\alpha$ being of type (A.4). For integer values, a matrix $e^{\beta'}_{\ \beta}$ is inverse to $e_{\beta'}^{\ \beta}$ (physicists call such matrices as vielbeins). Such a property holds for fractional constructions with the left Caputo fractional derivatives.

Similarly to $\overset{\alpha}{\underline{T}}M$, the above constructions can be used for definition of fractional vector bundle $\overset{\alpha}{\underline{E}}$ on M, when the fiber indices of bases run values $a', b', ... = n+1, n+2, ..., n+m.$

For a tangent bundle TM, using 1–forms as respective duals local vector bases, we can define the co–tangent bundle T^*M on M. Using corresponding definitions of fractional forms (A.4) and dubbing the for fractional differentials the above constructions, we can construct the fractional cotangent bundle $\overset{\alpha}{\underline{T}}{}^{*} M$ and consider generalizations for co–vector bundle $\overset{\alpha}{\underline{E}}{}^{*}$ on M. We omit details on such constructions (and possible higher order fractional tangent/vector generalizations, fractional osculator bundles etc) in this paper.

2.2 Fundamental geometric objects on fractional manifolds

Let us consider a fractional nonholonomic manifold $\overset{\alpha}{\mathbf{V}}$ defined by a quadruple $(\mathbf{V}, \overset{\alpha}{\mathcal{N}}, \overset{\alpha}{\mathbf{d}}, \overset{\alpha}{\mathbf{I}})$, where the fractional differential structure $\overset{\alpha}{\mathbf{d}}$ is stated by some (1) and (3) and the non–integer integral structure $\overset{\alpha}{\mathbf{I}}$ is given by rules of type (A.2). A "prime" integer manifold $\mathbf{V}$ is of integer dimension dim $\mathbf{V} = n+m, n \geq 2, m \geq 1$. Local coordinates on $\mathbf{V}$ are labeled in the form $u = (x, y)$, or $u^{\alpha} = (x^i, y^a)$, where indices $i, j, ... = 1, 2, ..., n$ are horizontal (h) ones and $a, b, ... = 1, 2, ..., m$ are vertical (v) ones. For some important examples, we have that $\mathbf{V} = TM$ is a tangent bundle, or $\mathbf{V} = \mathbf{E}$ is a vector bundle, on M, or $\mathbf{V}$ is a (semi–) Riemann manifold, with prescribed local (non–integrable) fibred structure. A nonholonomic manifold $\mathbf{V}$ is considered to be enabled with a non—integrable distribution defining a nonlinear connection as we explained in point 2 of section 1.2.

2.2.1 N–connections for fractional nonholonomic manifolds

A nonintegrable distribution $\overset{\alpha}{\mathcal{N}}$ for $\overset{\alpha}{\mathbf{V}}$ can be chosen in a form defining a nonlinear connection structure correspondingly adapted to chosen fractional calculus with $\overset{\alpha}{\mathbf{d}}$ and $\overset{\alpha}{\mathbf{I}}$.

Definition 2.1 *A nonlinear connection (N–connection)* $\overset{\alpha}{\mathbf{N}}$ *is defined by a Whitney sum of conventional h– and v–subspaces,* $\underline{h}\overset{\alpha}{\mathbf{V}}$ *and* $\underline{v}\overset{\alpha}{\mathbf{V}}$,

$$\overset{\alpha}{\underline{T}}\overset{\alpha}{\mathbf{V}} = \underline{h}\overset{\alpha}{\mathbf{V}} \oplus \underline{v}\overset{\alpha}{\mathbf{V}}, \qquad (5)$$

where the fractional tangent bundle $\overset{\alpha}{\underline{T}}\overset{\alpha}{\mathbf{V}}$ is constructed following the approach with the left Caputo fractional derivative chosen for the differential structure.

We note that a conventional splitting into h- and v-components depends on the type of chosen fractional left derivative. We underline some symbols if it is important to emphasize that the corresponding geometric objects are induced by the Caputo fractional derivative, but we shall omit "underling" if that will simplify the system of notation not resulting in ambiguities.

Nonholonomic manifolds with $\overset{\alpha}{\mathcal{N}}$ determined by a $\overset{\alpha}{\mathbf{N}}$ are called N–anholonomic fractional manifolds. In brief, we shall call them as fractional spaces (geometries/manifolds). Locally, a fractional N–connection is defined by its coefficients, $\overset{\alpha}{\mathbf{N}}=\{\ {}^{\alpha}N_i^a\}$, stated with respect to a local coordinate basis,

$$\overset{\alpha}{\mathbf{N}}=\ {}^{\alpha}N_i^a(u)(dx^i)^{\alpha}\otimes\overset{\alpha}{\underline{\partial}}_a, \tag{6}$$

see formulas (2) and (4).

N–connections are naturally considered in Finsler and Lagrange geometry, Einstein gravity, and various supersymmetric, noncommutative, quantum generalizations in modern (super) string/brane theories and geometric mechanics, see reviews of results in [14, 15, 11, 12, 13, 29, 30, 31].

Proposition 2.1 *A N–connection $\overset{\alpha}{\mathbf{N}}$ defines N–adapted (i.e. linearly depending on coefficients ${}^{\alpha}N_i^a$) fractional frame*

$$^{\alpha}\mathbf{e}_{\beta}=\left[\ {}^{\alpha}\mathbf{e}_j=\overset{\alpha}{\underline{\partial}}_j-\ {}^{\alpha}N_j^a\overset{\alpha}{\underline{\partial}}_a,\ {}^{\alpha}e_b=\overset{\alpha}{\underline{\partial}}_b\right] \tag{7}$$

and coframe

$$^{\alpha}\mathbf{e}^{\beta}=[\ {}^{\alpha}e^j=(dx^j)^{\alpha},\ {}^{\alpha}\mathbf{e}^b=(dy^b)^{\alpha}+\ {}^{\alpha}N_k^b(dx^k)^{\alpha}] \tag{8}$$

nonholonomic structures.

Proof. The corresponding nonholonomic integro–differential fractional structure is induced by the left Caputo derivative (A.1) and N–connection coefficients in (6). The nontrivial nonholonomy coefficients are computed ${}^{\alpha}W_{ib}^a=\overset{\alpha}{\underline{\partial}}_b\ {}^{\alpha}N_i^a$ and ${}^{\alpha}W_{ij}^a=\ {}^{\alpha}\Omega_{ji}^a=\ {}^{\alpha}\mathbf{e}_i\ {}^{\alpha}N_j^a-\ {}^{\alpha}\mathbf{e}_j\ {}^{\alpha}N_i^a$ (where ${}^{\alpha}\Omega_{ji}^a$ are the coefficients of the N–connection curvature) for

$$[\ {}^{\alpha}\mathbf{e}_{\alpha},\ {}^{\alpha}\mathbf{e}_{\beta}]=\ {}^{\alpha}\mathbf{e}_{\alpha}\ {}^{\alpha}\mathbf{e}_{\beta}-\ {}^{\alpha}\mathbf{e}_{\beta}\ {}^{\alpha}\mathbf{e}_{\alpha}=\ {}^{\alpha}W_{\alpha\beta}^{\gamma}\ {}^{\alpha}\mathbf{e}_{\gamma}.$$

For simplicity, in above formulas derived for (7) and (8), we omitted underlying of symbols of type ${}^{\alpha}\mathbf{e}_{\beta} = [\ {}^{\alpha}\mathbf{e}_j,\ {}^{\alpha}e_b]$ even such values are determined by fractional Caputo derivatives of type (A.1), which are underlined.

(End proof.) $\square$

2.2.2 N–adapted fractional metrics

A second fundamental geometric object on $\overset{\alpha}{\mathbf{V}}$, a metric $\overset{\alpha}{\mathbf{g}}$, can be defined similarly to (pseudo) Riemannian spaces of integer dimension but for a chosen fractional differential structure.

Definition 2.2 *A (fractional) metric structure* $\overset{\alpha}{\mathbf{g}} = \{\ {}^{\alpha}g_{\underline{\alpha\beta}}\}$ *is determined on a* $\overset{\alpha}{\mathbf{V}}$ *by a symmetric second rank tensor*

$$\overset{\alpha}{\mathbf{g}} = {}^{\alpha}g_{\underline{\gamma\beta}}(u)(du^{\underline{\gamma}})^{\alpha} \otimes (du^{\underline{\beta}})^{\alpha} \qquad (9)$$

for a tensor product of fractional coordinate co–bases (4).

For N–adapted constructions, it is important to introduce and prove:

Claim 2.1 *Any fractional metric* $\overset{\alpha}{\mathbf{g}}$ *can be represented equivalently as a distinguished metric structure (d–metric),* $\overset{\alpha}{\mathbf{g}} = [\ {}^{\alpha}g_{kj},\ {}^{\alpha}g_{cb}]$, *which is N–adapted to splitting (5),*

$$\overset{\alpha}{\mathbf{g}} = {}^{\alpha}g_{kj}(x,y)\ {}^{\alpha}e^{k} \otimes {}^{\alpha}e^{j} + {}^{\alpha}g_{cb}(x,y)\ {}^{\alpha}\mathbf{e}^{c} \otimes {}^{\alpha}\mathbf{e}^{b}, \qquad (10)$$

where fractional N–elongated bases ${}^{\alpha}\mathbf{e}^{\beta} = [\ {}^{\alpha}e^{j},\ {}^{\alpha}\mathbf{e}^{b}]$ *are defined as in (7).*

Proof. For coefficients of metric (9), we consider parametrization

$${}^{\alpha}g_{\underline{\alpha\beta}} = \left[\begin{array}{cc} {}^{\alpha}\underline{g}_{ij} = {}^{\alpha}g_{ij} + {}^{\alpha}N_i^a\ {}^{\alpha}N_j^b\ {}^{\alpha}g_{ab} & {}^{\alpha}\underline{g}_{ib} = {}^{\alpha}N_i^e\ {}^{\alpha}g_{be} \\ {}^{\alpha}\underline{g}_{aj} = {}^{\alpha}N_i^e\ {}^{\alpha}g_{be} & {}^{\alpha}\underline{g}_{ab} \end{array} \right], \qquad (11)$$

for ${}^{\alpha}g_{\underline{\alpha\beta}} = {}^{\alpha}\underline{g}_{\alpha\beta}$. We introduce the vielbeins

$$\mathbf{e}_{\alpha}^{\ \underline{\alpha}} = \left[\begin{array}{cc} e_i^{\ \underline{i}} = \delta_i^{\underline{i}} & e_i^{\ \underline{a}} = {}^{\alpha}N_i^b\ \delta_b^{\underline{a}} \\ e_a^{\ \underline{i}} = 0 & e_a^{\ \underline{a}} = \delta_a^{\underline{a}} \end{array} \right],\ \mathbf{e}^{\alpha}_{\ \underline{\alpha}} = \left[\begin{array}{cc} e^i_{\ \underline{i}} = \delta^i_{\underline{i}} & e^b_{\ \underline{i}} = -\ {}^{\alpha}N_k^b\ \delta_{\underline{i}}^k \\ e^i_{\ \underline{a}} = 0 & e^a_{\ \underline{a}} = \delta^a_{\underline{a}} \end{array} \right], \qquad (12)$$

where $\delta^i_{\underline{i}}$ is the Kronecher symbol, and define nonholonomic frames

$${}^{\alpha}\mathbf{e}_{\beta} = \mathbf{e}_{\beta}^{\ \underline{\beta}}\ \partial_{\underline{\beta}} \text{ and } \quad {}^{\alpha}\mathbf{e}^{\alpha} = \mathbf{e}^{\beta}_{\ \underline{\beta}}(du^{\underline{\beta}})^{\alpha},$$

which are N–adapted frames, respectively, of type (7) and (8). Re–grouping the coefficients, we get the formula (10). $\square$

2.2.3 Distinguished fractional connections

Linear connections on fractional $\overset{\alpha}{\mathbf{V}}$ may be adapted to the N–connection structure as for the integer dimensions.

Definition 2.3 *A distinguished connection (d–connection)* $\overset{\alpha}{\mathbf{D}}$ *on* $\overset{\alpha}{\mathbf{V}}$ *is a linear connection preserving under parallel transports the Whitney sum (5).*

A covariant fractional calculus on nonholonomic manifolds can be developed following the formalism of fractional differential forms. For a fractional d–connection $\overset{\alpha}{\mathbf{D}}$, we can introduce a N–adapted differential 1–form of type (A.4)

$$ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta} = {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta\gamma}\ {}^{\alpha}\mathbf{e}^{\gamma}, \tag{13} $$

with the coefficients defined with respect to (8) and (7) and parametrized the form ${}^{\alpha}\mathbf{\Gamma}^{\gamma}{}_{\tau\beta} = \left({}^{\alpha}L^{i}_{jk},\ {}^{\alpha}L^{a}_{bk},\ {}^{\alpha}C^{i}_{jc},\ {}^{\alpha}C^{a}_{bc} \right).$

We also consider that the absolute fractional differential ${}^{\alpha}\mathbf{d} = {}_{1x}\overset{\alpha}{d}_{x} + {}_{1y}\overset{\alpha}{d}_{y}$ is a N–adapted fractional operator ${}^{\alpha}\mathbf{d} := {}^{\alpha}\mathbf{e}^{\beta}\ {}^{\alpha}\mathbf{e}_{\beta}$ defined by exterior h- and v–derivatives of type (A.3), when

$$ {}_{1x}\overset{\alpha}{d}_{x} := (dx^{i})^{\alpha} \quad {}_{1x}\overset{\alpha}{\underline{\partial}}_{i} = {}^{\alpha}e^{j}\ {}^{\alpha}\mathbf{e}_{j} \text{ and } {}_{1y}\overset{\alpha}{d}_{y} := (dy^{a})^{\alpha} \quad {}_{1x}\overset{\alpha}{\underline{\partial}}_{a} = {}^{\alpha}\mathbf{e}^{b}\ {}^{\alpha}e_{b}. $$

Definition 2.4 *The torsion of a fractional d–connection* $\overset{\alpha}{\mathbf{D}} = \{ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta\gamma} \}$ *is*

$$ {}^{\alpha}\mathcal{T}^{\tau} \doteqdot \overset{\alpha}{\mathbf{D}}\ {}^{\alpha}\mathbf{e}^{\tau} = {}^{\alpha}\mathbf{d}\ {}^{\alpha}\mathbf{e}^{\tau} + {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta} \wedge {}^{\alpha}\mathbf{e}^{\beta}. \tag{14} $$

Following an explicit fractional (and N–adapted) differential form calculus with respect to (8), we prove:

Theorem 2.1 *Locally, the fractional torsion* ${}^{\alpha}\mathcal{T}^{\tau}$ *(14) is characterized by its coefficients (d–torsion)*

$$ \begin{aligned} {}^{\alpha}T^{i}{}_{jk} &= {}^{\alpha}L^{i}{}_{jk} - {}^{\alpha}L^{i}{}_{kj},\ \ {}^{\alpha}T^{i}{}_{ja} = -\,{}^{\alpha}T^{i}{}_{aj} = {}^{\alpha}C^{i}{}_{ja},\ \ {}^{\alpha}T^{a}{}_{ji} = {}^{\alpha}\Omega^{a}{}_{ji}, \\ {}^{\alpha}T^{a}{}_{bi} &= -\,{}^{\alpha}T^{a}{}_{ib} = {}^{\alpha}e_{b}\ {}^{\alpha}N^{a}_{i} - {}^{\alpha}L^{a}{}_{bi},\ \ {}^{\alpha}T^{a}{}_{bc} = {}^{\alpha}C^{a}{}_{bc} - {}^{\alpha}C^{a}{}_{cb}. \end{aligned} \tag{15} $$

For integer α, we get the same formulas as in [11, 12, 13, 14]. This is possible if we consider on $\overset{\alpha}{\mathbf{V}}$ a differential structure which locally can be induced by the left Caputo fractional derivatives and associated differentials.

Definition 2.5 *The curvature of a fractional* $\overset{\alpha}{\mathbf{D}} = \{ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta\gamma}\}$ *is*

$$ {}^{\alpha}\mathcal{R}^{\tau}{}_{\beta} \doteqdot \overset{\alpha}{\mathbf{D}}\ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta} = {}^{\alpha}\mathbf{d}\ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta} - {}^{\alpha}\mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\gamma} = {}^{\alpha}\mathbf{R}^{\tau}{}_{\beta\gamma\delta}\ {}^{\alpha}\mathbf{e}^{\gamma} \wedge {}^{\alpha}\mathbf{e} \quad (16)$$

A straightforward fractional differential form calculus for (13) gives a proof of

Theorem 2.2 *Locally, the fractional curvature* ${}^{\alpha}\mathcal{R}^{\tau}{}_{\beta}$ *(16) is characterized by its coefficients (d–curvature)*

$$\begin{aligned}
{}^{\alpha}R^{i}{}_{hjk} &= {}^{\alpha}\mathbf{e}_{k}\ {}^{\alpha}L^{i}{}_{hj} - {}^{\alpha}\mathbf{e}_{j}\ {}^{\alpha}L^{i}{}_{hk} \\
&\quad + {}^{\alpha}L^{m}_{hj}\ {}^{\alpha}L^{i}{}_{mk} - {}^{\alpha}L^{m}_{hk}\ {}^{\alpha}L^{i}{}_{mj} - {}^{\alpha}C^{i}{}_{ha}\ {}^{\alpha}\Omega^{a}{}_{kj}, \\
{}^{\alpha}R^{a}{}_{bjk} &= {}^{\alpha}\mathbf{e}_{k}\ {}^{\alpha}L^{a}{}_{bj} - {}^{\alpha}\mathbf{e}_{j}\ {}^{\alpha}L^{a}{}_{bk} \\
&\quad + {}^{\alpha}L^{c}{}_{bj}\ {}^{\alpha}L^{a}{}_{ck} - {}^{\alpha}L^{c}{}_{bk}\ {}^{\alpha}L^{a}{}_{cj} - {}^{\alpha}C^{a}{}_{bc}\ {}^{\alpha}\Omega^{c}{}_{kj}, \\
{}^{\alpha}R^{i}{}_{jka} &= {}^{\alpha}e_{a}\ {}^{\alpha}L^{i}{}_{jk} - {}^{\alpha}D_{k}\ {}^{\alpha}C^{i}{}_{ja} + {}^{\alpha}C^{i}{}_{jb}T^{b}{}_{ka}, \qquad (17)\\
{}^{\alpha}R^{c}{}_{bka} &= {}^{\alpha}e_{a}\ {}^{\alpha}L^{c}{}_{bk} - {}^{\alpha}D_{k}\ {}^{\alpha}C^{c}{}_{ba} + {}^{\alpha}C^{c}{}_{bd}\ {}^{\alpha}T^{c}{}_{ka}, \\
{}^{\alpha}R^{i}{}_{jbc} &= {}^{\alpha}e_{c}\ {}^{\alpha}C^{i}{}_{jb} - {}^{\alpha}e_{b}\ {}^{\alpha}C^{i}{}_{jc} + {}^{\alpha}C^{h}{}_{jb}\ {}^{\alpha}C^{i}{}_{hc} - {}^{\alpha}C^{h}{}_{jc}\ {}^{\alpha}C^{i}{}_{hb}, \\
{}^{\alpha}R^{a}{}_{bcd} &= {}^{\alpha}e_{d}\ {}^{\alpha}C^{a}{}_{bc} - {}^{\alpha}e_{c}\ {}^{\alpha}C^{a}{}_{bd} + {}^{\alpha}C^{e}{}_{bc}\ {}^{\alpha}C^{a}{}_{ed} - {}^{\alpha}C^{e}{}_{bd}\ {}^{\alpha}C^{a}{}_{ec}.
\end{aligned}$$

Formulas (15) and (17) encode integro–differential nonholonomic structures modeling certain types of fractional differential geometric models. For integer dimensions, on vector/tangent bundles, such constructions are typical ones for Lagrange–Finsler geometry [14] and various types generalizations in modern geometry and gravity [30, 13, 12, 31].

Contracting respectively the components of (17), we can prove

Proposition 2.2 *The fractional Ricci tensor* ${}^{\alpha}\mathcal{R}ic = \{ {}^{\alpha}\mathbf{R}_{\alpha\beta} \doteqdot {}^{\alpha}\mathbf{R}^{\tau}{}_{\alpha\beta\tau}\}$ *is characterized by h- v–components, i.e. d–tensors,*

$$ {}^{\alpha}R_{ij} \doteqdot {}^{\alpha}R^{k}{}_{ijk},\quad {}^{\alpha}R_{ia} \doteqdot - {}^{\alpha}R^{k}{}_{ika},\quad {}^{\alpha}R_{ai} \doteqdot {}^{\alpha}R^{b}{}_{aib},\quad {}^{\alpha}R_{ab} \doteqdot {}^{\alpha}R^{c}{}_{abc}. \quad (18)$$

It is obvious that the fractional Ricci tensor ${}^{\alpha}\mathbf{R}_{\alpha\beta}$ is not symmetric for arbitrary fractional d–connections.

For a fractional d–metric structure (10), we can introduce:

Definition 2.6 *The scalar curvature of a fractional d–connection* $\overset{\alpha}{\mathbf{D}}$ *is*

$$\begin{aligned}
{}^{\alpha}_{s}\mathbf{R} &\doteqdot {}^{\alpha}\mathbf{g}^{\tau\beta}\ {}^{\alpha}\mathbf{R}_{\tau\beta} = {}^{\alpha}R + {}^{\alpha}S, \qquad (19)\\
{}^{\alpha}R &= {}^{\alpha}g^{ij}\ {}^{\alpha}R_{ij},\quad {}^{\alpha}S = {}^{\alpha}g^{ab}\ {}^{\alpha}R_{ab},
\end{aligned}$$

defined by a sum the h– and v–components of (18) and contractions with the inverse coefficients to a d–metric (10).

Proposition 2.3 -*Definition:* *The Einstein tensor* ${}^{\alpha}\mathcal{E}ns = \{ {}^{\alpha}\mathbf{G}_{\alpha\beta}\}$ *for a fractional d–connection* $\overset{\alpha}{\mathbf{D}}$ *is computed in standard form*

$$ {}^{\alpha}\mathbf{G}_{\alpha\beta} := {}^{\alpha}\mathbf{R}_{\alpha\beta} - \frac{1}{2}\ {}^{\alpha}\mathbf{g}_{\alpha\beta}\ {}_{s}^{\alpha}\mathbf{R}. \qquad (20)$$

Such a tensor can be used for various fractional generalizations of the Einstein and Lagrange–Finsler gravity models from [11, 12, 13, 14]. It should be emphasized that variants of fractional Ricci and Einstein tensor were considered in [28, 32], respectively, for generalized fractional Rimann–Finsler and Einstein spaces but with RL fractional derivatives. Technically, it is a very cumbersome task to find solutions of such sophisticate integro–differential equations and study possible physical implications. In our approach, working with the left Caputo fractional derivative and by corresponding nonholonomic transforms, we can separate the equations in fractional equations in such a form that the resulting systems of partial differential and integral equations can integrated exactly in very general form similarly to the integer cases outlined for different models of gravity theory in [33, 11, 12, 13] and, for nonholonomic Ricci flows and applications to physics, in [34, 35, 35, 37, 38, 10].

2.2.4 The fractional canonical d–connection and Levi–Civita connection

There are an infinite number of fractional d–connections $\overset{\alpha}{\mathbf{D}}$ on $\overset{\alpha}{\mathbf{V}}$. For applications in modern geometry and physics, a special interest present subclasses of such linear connections which are metric compatible with a metric structure, i.e. $\overset{\alpha}{\mathbf{D}}\,({}^{\alpha}\mathbf{g}) = 0$, with more special cases when $\overset{\alpha}{\mathbf{D}}$ is completely and uniquely determined by ${}^{\alpha}\mathbf{g}$ and $\overset{\alpha}{\mathbf{N}}$ following certain well–defined geometric/physical principles.

Theorem 2.3 *There is a unique canonical fractional d–connection* ${}^{\alpha}\widehat{\mathbf{D}} = \{ {}^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left({}^{\alpha}\widehat{L}^{i}_{jk},\ {}^{\alpha}\widehat{L}^{a}_{bk},\ {}^{\alpha}\widehat{C}^{i}_{jc},\ {}^{\alpha}\widehat{C}^{a}_{bc}\right)\}$ *which is compatible with the metric structure,* ${}^{\alpha}\widehat{\mathbf{D}}\ ({}^{\alpha}\mathbf{g}) = 0$, *and satisfies the conditions* ${}^{\alpha}\widehat{T}^{i}{}_{jk} = 0$ *and* ${}^{\alpha}\widehat{T}^{a}{}_{bc} = 0.$

Proof. It follows from explicit formulas for coefficients of (10) and

$$
\begin{aligned}
{}^{\alpha}\widehat{L}^{i}_{jk} &= \frac{1}{2}\,{}^{\alpha}g^{ir}\left({}^{\alpha}\mathbf{e}_{k}\,{}^{\alpha}g_{jr} + {}^{\alpha}\mathbf{e}_{j}\,{}^{\alpha}g_{kr} - {}^{\alpha}\mathbf{e}_{r}\,{}^{\alpha}g_{jk}\right), \qquad (21)\\
{}^{\alpha}\widehat{L}^{a}_{bk} &= {}^{\alpha}e_{b}({}^{\alpha}N^{a}_{k}) + \\
&\frac{1}{2}\,{}^{\alpha}g^{ac}\left({}^{\alpha}\mathbf{e}_{k}\,{}^{\alpha}g_{bc} - {}^{\alpha}g_{dc}\,{}^{\alpha}e_{b}\,{}^{\alpha}N^{d}_{k} - {}^{\alpha}g_{db}\,{}^{\alpha}e_{c}\,{}^{\alpha}N^{d}_{k}\right),\\
{}^{\alpha}\widehat{C}^{i}_{jc} &= \frac{1}{2}\,{}^{\alpha}g^{ik}\,{}^{\alpha}e_{c}\,{}^{\alpha}g_{jk},\\
{}^{\alpha}\widehat{C}^{a}_{bc} &= \frac{1}{2}\,{}^{\alpha}g^{ad}\left({}^{\alpha}e_{c}\,{}^{\alpha}g_{bd} + {}^{\alpha}e_{c}\,{}^{\alpha}g_{cd} - {}^{\alpha}e_{d}\,{}^{\alpha}g_{bc}\right).
\end{aligned}
$$

Introducing the values (21) into formulas (15) we obtain that $\widehat{T}^{i}{}_{jk} = 0$ and $\widehat{T}^{a}{}_{bc} = 0$, but $\widehat{T}^{i}{}_{ja}, \widehat{T}^{a}{}_{ji}$ and $\widehat{T}^{a}{}_{bi}$ are not zero, that the metricity conditions are satisfied in component form. □

On a fractional nonholonomic $\overset{\alpha}{\mathbf{V}}$, the Levi–Civita connection ${}^{\alpha}\nabla = \{{}^{\alpha}\Gamma^{\gamma}{}_{\alpha\beta}\}$ can be defined in standard from by using the fractional Caputo left derivatives acting correspondingly on the coefficients of a fractional metric (9). Such a geometric object is not adapted to the N–connection splitting (5). As a consequence of nonholonomic structure, it is preferred to work on $\overset{\alpha}{\mathbf{V}}$ with ${}^{\alpha}\widehat{\mathbf{D}} = \{{}^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\tau\beta}\}$ instead of ${}^{\alpha}\nabla$. Even ${}^{\alpha}\widehat{\mathbf{D}}$ has a nontrivial d–torsion, such an object is very different from a similar one, for instance, in "integer" Einstein–Cartan gravity when additional gravitational equations have to be introduces for the nontrivial torsion components. In our case, the canonical ${}^{\alpha}\widehat{\mathcal{T}}^{\tau}$ (14) is nonholonomically induced, via fractional integral and derivative operations, in a unique form, by some off–diagonal coefficients of metric field.

Let us parametrize the coefficients of ${}^{\alpha}\nabla$ (for integer α, it is uniquely derived from the conditions ${}_{\shortmid}\mathcal{T} = 0$ and $\triangledown g = 0$) in the form

$$
\begin{aligned}
{}^{\alpha}\Gamma^{\alpha}_{\beta\gamma} &= \left({}^{\alpha}L^{i}_{jk},\ {}^{\alpha}L^{a}_{jk},\ {}^{\alpha}L^{i}_{bk},\ {}^{\alpha}L^{a}_{bk},\ {}^{\alpha}C^{i}_{jb},\ {}^{\alpha}C^{a}_{jb},\ {}^{\alpha}C^{i}_{bc},\ {}^{\alpha}C^{a}_{bc}\right),\\
\text{where}\quad & {}^{\alpha}\triangledown_{{}^{\alpha}\mathbf{e}_{k}}({}^{\alpha}\mathbf{e}_{j}) = {}^{\alpha}L^{i}_{jk}\,{}^{\alpha}\mathbf{e}_{i} + {}^{\alpha}L^{a}_{jk}\,{}^{\alpha}e_{a},\\
& {}^{\alpha}\triangledown_{{}^{\alpha}\mathbf{e}_{k}}({}^{\alpha}e_{b}) = {}^{\alpha}L^{i}_{bk}\,{}^{\alpha}\mathbf{e}_{i} + {}^{\alpha}L^{a}_{bk}\,{}^{\alpha}e_{a},\\
& {}^{\alpha}\triangledown_{{}^{\alpha}e_{b}}({}^{\alpha}\mathbf{e}_{j}) = {}^{\alpha}C^{i}_{jb}\,{}^{\alpha}\mathbf{e}_{i} + {}^{\alpha}C^{a}_{jb}\,{}^{\alpha}e_{a},\\
& {}^{\alpha}\triangledown_{{}^{\alpha}e_{c}}({}^{\alpha}e_{b}) = {}^{\alpha}C^{i}_{bc}\,{}^{\alpha}\mathbf{e}_{i} + {}^{\alpha}C^{a}_{bc}\,{}^{\alpha}e_{a}.
\end{aligned}
$$

Following a straightforward fractional coefficient computation, we can prove

Corollary 2.1 *With respect to N–adapted fractional bases (7) and (8), the coefficients of the fractional Levi–Civita and canonical d–connection satisfy*

the distorting relations

$$ {}^{\alpha}\Gamma^{\gamma}_{\ \alpha\beta} = {}^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta} + {}^{\alpha}Z^{\gamma}_{\ \alpha\beta} \quad (22) $$

where the explicit components of distortion tensor ${}_{\shortmid}Z^{\gamma}_{\ \alpha\beta}$ *are computed*

$$
\begin{aligned}
{}^{\alpha}Z^{i}_{jk} &= 0, \quad {}^{\alpha}Z^{a}_{jk} = -\ {}^{\alpha}C^{i}_{jb}\ {}^{\alpha}g_{ik}\ {}^{\alpha}g^{ab} - \frac{1}{2}\ {}^{\alpha}\Omega^{a}_{jk}, \\
{}^{\alpha}Z^{i}_{bk} &= \frac{1}{2}\ {}^{\alpha}\Omega^{c}_{jk}\ {}^{\alpha}g_{cb}\ {}^{\alpha}g^{ji} - \frac{1}{2}(\delta^{i}_{j}\delta^{h}_{k} -\ {}^{\alpha}g_{jk}\ {}^{\alpha}g^{ih})\ {}^{\alpha}C^{j}_{hb}, \\
{}^{\alpha}Z^{a}_{bk} &= \frac{1}{2}(\delta^{a}_{c}\delta^{b}_{d} +\ {}^{\alpha}g_{cd}\ {}^{\alpha}g^{ab})\left[\ {}^{\alpha}L^{c}_{bk} -\ {}^{\alpha}e_{b}(\ {}^{\alpha}N^{c}_{k})\right], \\
{}^{\alpha}Z^{i}_{kb} &= \frac{1}{2}\ {}^{\alpha}\Omega^{a}_{jk}\ {}^{\alpha}g_{cb}\ {}^{\alpha}g^{ji} + \frac{1}{2}(\delta^{i}_{j}\delta^{h}_{k} -\ {}^{\alpha}g_{jk}\ {}^{\alpha}g^{ih})\ {}^{\alpha}C^{j}_{hb}, \\
{}^{\alpha}Z^{a}_{jb} &= -\frac{1}{2}(\delta^{a}_{c}\delta^{d}_{b} -\ {}^{\alpha}g_{cb}\ {}^{\alpha}g^{ad})\left[\ {}^{\alpha}L^{c}_{dj} -\ {}^{\alpha}e_{d}(\ {}^{\alpha}N^{c}_{j})\right], \qquad (23) \\
{}^{\alpha}Z^{a}_{bc} &= 0, \\
{}^{\alpha}Z^{i}_{ab} &= -\frac{{}^{\alpha}g^{ij}}{2}\left\{\left[\ {}^{\alpha}L^{c}_{aj} -\ {}^{\alpha}e_{a}(\ {}^{\alpha}N^{c}_{j})\right]\ {}^{\alpha}g_{cb} \right. \\
&\left. + \left[\ {}^{\alpha}L^{c}_{bj} -\ {}^{\alpha}e_{b}(\ {}^{\alpha}N^{c}_{j})\right]\ {}^{\alpha}g_{ca}\right\}.
\end{aligned}
$$

We emphasize that there are not simple relations of type (22) and (23) if the fractional integro–differential structure would be not elaborated in N–adapted form for the left Caputo derivative. For the fractional RL derivatives, it is not possible to introduce N–anholonomic distributions when the formulas would preserve a maximal similarity with the integer nonholonomic case.

3 Perelman Type Fractional Functionals

The goal of this section is to show that there is a fractional integro–differential calculus admitting generalizations of the Hamilton–Perelman Ricci flow evolution theory. Proofs are simplified for correspondingly defined nonholonomic fractional distributions.

3.1 On (non) holonomic Ricci flows

For Riemannian spaces of integer dimension, the Grisha Perelman's fundamental idea was to prove that the Ricci flow is not only a gradient flow but, introducing two Lyapunov type functionals, can be defined also as a dynamical system on the spaces of Riemannian metrics.

The Ricci flow equation was postulated by R. Hamilton [1] as an evolution equation[5]

$$\frac{\partial g_{\alpha\beta}(\chi)}{\partial \chi} = -2\ R_{\alpha\beta}(\chi) \tag{24}$$

for a set of Riemannian metrics $g_{\alpha\beta}(\chi)$ and corresponding Ricci tensors $R_{\alpha\beta}(\chi)$, derived for corresponding set of Levi–Civita connections $\nabla(\chi)$, all parametrized by a real parameter χ.

The Perelman's functionals were introduced for Ricci flows of Riemannian metrics. For the Levi–Civita- connection ∇ defined by a metric $\mathbf{g}$, such fundamental functionals are written in the form

$$\begin{aligned} {}_{\shortmid}\mathcal{F}(f) &= \int\limits_{\mathbf{V}} \left({}_{\shortmid}R + |\nabla f|^2 \right) e^{-f}\ dV, \qquad (25) \\ {}_{\shortmid}\mathcal{W}(f,\tau) &= \int\limits_{\mathbf{V}} \left[\tau \left({}_{\shortmid}R + |\nabla f| \right)^2 + f - \frac{n+m}{2} \right] \mu\ dV, \end{aligned}$$

where dV is the volume form of $\mathbf{g}$, integration is taken over compact $\mathbf{V}$ and ${}_{\shortmid}R$ is the scalar curvature computed for ∇. For a flow parameter $\tau > 0$, we have $\int_{\mathbf{V}} \mu dV = 1$ when $\mu = (4\pi\tau)^{-(n+m)/2}\, e^{-f}$.

In our works [8, 9, 10, 34, 35, 36], we proved that nonholonomic Ricci flows of the Lagrange–Finsler geometries and various generalizations with nonsymmetric metics, noncommutative structures etc, can be modelled as constrained structures on N–anholonomic Riemannian spaces. The main conclusion was that following a N–adapted formalism, the Ricci flow theory can be extended for non–Riemannian geometries.

3.2 Fractional functionals for nonholonomic Ricci flows

The functional approach can be redefined for N–anholonomic manifolds, for our purposes, modeled as fractional spaces $\overset{\alpha}{\mathbf{V}}$. Fractional flows are considered with fractional derivative on parameters. In N–adapted form, we follow the methods from [8, 9] extended for fractional derivatives.

Claim 3.1 *For fractional nonholonomic geometries defined by the canonical*

[5] for our further purposes, it is convenient to use a different system of denotations than those considered by R. Hamilton or Grisha Perelman on holonomic Riemannian spaces of integer dimensions

d–connection ${}^{\alpha}\widehat{\mathbf{D}}$, *the fractional versions of Perelman's functionals are*

$$
\begin{aligned}
{}^{\alpha}\widehat{\mathcal{F}}({}^{\alpha}\mathbf{g}, \overset{\alpha}{\mathbf{N}}, {}^{\alpha}\widehat{f}) &= \int\limits_{{}^{\alpha}\mathbf{V}} ({}^{\alpha}R + {}^{\alpha}S + |{}^{\alpha}\widehat{\mathbf{D}}\,{}^{\alpha}\widehat{f}|^2) e^{-\,{}^{\alpha}\widehat{f}}\,{}^{\alpha}d\,{}^{\alpha}V, \quad (26) \\
{}^{\alpha}\widehat{\mathcal{W}}({}^{\alpha}\mathbf{g}, \overset{\alpha}{\mathbf{N}}, {}^{\alpha}\widehat{f}, {}^{\alpha}\tau) &= \int\limits_{{}^{\alpha}\mathbf{V}} [\,{}^{\alpha}\widehat{\tau}({}^{\alpha}R + {}^{\alpha}S + |{}^{\alpha}_{h}D\,{}^{\alpha}\widehat{f}| + |{}^{\alpha}_{v}D\,{}^{\alpha}\widehat{f}|)^2 \\
&\quad + {}^{\alpha}\widehat{f} - \frac{n+m}{2}]\,{}^{\alpha}\widehat{\mu}\,{}^{\alpha}d\,{}^{\alpha}V, \quad (27)
\end{aligned}
$$

where ${}^{\alpha}d\,{}^{\alpha}V$ *is the volume fractional form of* ${}^{\alpha}\mathbf{g}$ *(10),* ${}^{\alpha}R$ *and* ${}^{\alpha}S$ *are respectively the h- and v–components of the curvature scalar (19) of* ${}^{\alpha}\widehat{\mathbf{D}}$, *for* ${}^{\alpha}\widehat{\mathbf{D}}_{\beta} = ({}^{\alpha}D_i, {}^{\alpha}D_a)$, *or* ${}^{\alpha}\widehat{\mathbf{D}} = ({}^{h}D, {}^{v}D)$, $\left|{}^{\alpha}\widehat{\mathbf{D}}\,{}^{\alpha}\widehat{f}\right|^2 = \left|{}^{\alpha}_{h}D\,{}^{\alpha}\widehat{f}\right|^2 + \left|{}^{\alpha}_{v}D\,{}^{\alpha}\widehat{f}\right|^2$, *and* ${}^{\alpha}\widehat{f}$ *satisfies* $\int_{{}^{\alpha}\mathbf{V}} {}^{\alpha}\widehat{\mu}\,{}^{\alpha}d\,{}^{\alpha}V = 1$ *for* ${}^{\alpha}\widehat{\mu} = (4\pi\,\tau)^{-(n+m)/2}\,e^{-\,{}^{\alpha}\widehat{f}}$ *and fractional flow parameter* $\tau > 0$.

Proof. Formulas (25) can be rewritten for some fractional functions ${}^{\alpha}\widehat{f}$ and ${}^{\alpha}f$ when

$$
\begin{aligned}
({}^{\alpha}_{\shortmid}R + |{}^{\alpha}\nabla\,{}^{\alpha}f|^2) e^{-\,{}^{\alpha}f} &= ({}^{\alpha}R + {}^{\alpha}S + \left|{}^{\alpha}_{h}D\,{}^{\alpha}\widehat{f}\right|^2 + \left|{}^{\alpha}_{v}D\,{}^{\alpha}\widehat{f}\right|^2) e^{-\,{}^{\alpha}\widehat{f}} \\
&\quad + {}^{\alpha}\Phi
\end{aligned}
$$

for a re–scaling of fractional parameter $\tau \to \widehat{\tau}$ with

$$
\begin{aligned}
&\left[{}^{\alpha}\tau\left({}^{\alpha}_{\shortmid}R + |{}^{\alpha}\nabla\,{}^{\alpha}f|\right)^2 + {}^{\alpha}f - \frac{n+m}{2})\right]\,{}^{\alpha}\mu = \\
&\left[{}^{\alpha}\widehat{\tau}\left({}^{\alpha}R + {}^{\alpha}S + \left|{}^{\alpha}_{h}D\widehat{f}\right| + \left|{}^{\alpha}_{v}D\widehat{f}\right|\right)^2 + {}^{\alpha}\widehat{f} - \frac{n+m}{2}\right]\,{}^{\alpha}\widehat{\mu} + {}^{\alpha}\Phi_1,
\end{aligned}
$$

for some ${}^{\alpha}\Phi$ and ${}^{\alpha}\Phi_1$ for which $\int\limits_{{}^{\alpha}\mathbf{V}} {}^{\alpha}\Phi\,{}^{\alpha}d\,{}^{\alpha}V = 0$ and $\int\limits_{{}^{\alpha}\mathbf{V}} {}^{\alpha}\Phi_1\,{}^{\alpha}d\,{}^{\alpha}V = 0$. $\square$

For proofs of the Main Results in section 4, the next lemma will be important.

Lemma 3.1 *The first N–adapted fractional variations of (26) are given by*

$$\delta\ {}^{\alpha}\widehat{\mathcal{F}}(v_{ij}, v_{ab},\ {}^{h}f,\ {}^{v}f) = \qquad (28)$$
$$\int_{\mathbf{V}} \{[-v_{ij}(\ {}^{\alpha}R_{ij} +\ {}^{\alpha}D_i\ {}^{\alpha}D_j\ {}^{\alpha}\widehat{f}) + (\frac{{}^{h}v}{2} -\ {}^{h}f)(2\ {}^{h}\Delta\widehat{f} - |\ {}^{h}D\ \widehat{f}|) +\ {}^{\alpha}R]$$
$$+[-v_{ab}(\ {}^{\alpha}R_{ab} +\ {}^{\alpha}D_a\ {}^{\alpha}D_b\ {}^{\alpha}\widehat{f}) + (\frac{{}^{v}v}{2} -\ {}^{v}f)\left(2\ {}^{v}\Delta\widehat{f} - |\ {}^{v}D\ \widehat{f}|\right)$$
$$+\ {}^{\alpha}S]\}e^{-\ {}^{\alpha}\widehat{f}}\ {}^{\alpha}d\ {}^{\alpha}V,$$

where ${}^{h}\Delta =\ {}^{\alpha}D_i\ {}^{\alpha}D^i$ *and* ${}^{v}\Delta =\ {}^{\alpha}D_a\ {}^{\alpha}D^a, \widehat{\Delta} =\ {}^{h}\Delta +\ {}^{v}\Delta,$ *and* ${}^{h}v = {}^{\alpha}g^{ij}v_{ij},\ {}^{v}v =\ {}^{\alpha}g^{ab}v_{ab};$ *for h–variation* ${}^{h}\delta\ {}^{\alpha}g_{ij} = v_{ij}$, *v-variation* ${}^{v}\delta\ {}^{\alpha}g_{ab} = v_{ab}$ *and variations* ${}^{h}\delta\ {}^{\alpha}\widehat{f} =\ {}^{h}f,\ {}^{v}\delta\ {}^{\alpha}\widehat{f} =\ {}^{v}f.$

Proof. We fix a N–connection structure $\overset{\alpha}{\mathbf{N}}$ for a fractional metric ${}^{\alpha}\mathbf{g}$ (10). Then we follow a N–adapted fractional calculus similar to that for Perelman's Lemma in [2]. We omit details given, for instance, in the proof for integer configurations in [5], see there Lemma 1.5.2. □

4 Fractional Hamilton's Evolution Equations

In this section, we formulate the main results of this paper, on fractional Ricci flow theory: we sketch the proofs that evolution of fractional geometries can be derived by variation of generalized Perelman functionals and show that a statistical analogy can be provided to such fractional flow processes. For integer dimensions, such constructions model holonomic Ricci flows of (pseudo) Riemannian and Kähler geometries [2, 5, 6, 7].

A heuristic approach to develop a fractional Ricci flow theory is to take the equations

$$\frac{\partial}{\partial\chi}g_{\underline{\alpha}\underline{\beta}} = -2\ {}_{\shortmid}R_{\underline{\alpha}\underline{\beta}} + \frac{2r}{5}g_{\underline{\alpha}\underline{\beta}}, \qquad (29)$$

describing normalized (holonomic) Ricci flows with respect to a coordinate base $\partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$.[6] In (29), the normalizing factor $r = \int\ {}_{\shortmid}RdV/dV$ is introduced in order to preserve the volume V; ${}_{\shortmid}R_{\underline{\alpha}\underline{\beta}}$ and ${}_{\shortmid}R = g^{\underline{\alpha}\underline{\beta}}\ {}_{\shortmid}R_{\underline{\alpha}\underline{\beta}}$ are computed for the Levi–Civita connection ∇. Then we change the geometric

[6]In this integer case, we underline the indices with respect to the coordinate bases in order to distinguish them from those defined with respect to the 'N–elongated' local bases (7) and (8).

objects (tensors, derivatives and parameter) into fractional ones, and obtain a non–integer generalization of Hamilton's equations,

$$
\begin{aligned}
{}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}g_{ij} &= 2[\ {}^{\alpha}N_i^a\ {}^{\alpha}N_j^b\ (\ {}_{\shortmid}^{\alpha}R_{ab} -\ {}^{\alpha}\lambda\ {}^{\alpha}g_{ab}) -\ {}_{\shortmid}^{\alpha}R_{ij} +\ {}^{\alpha}\lambda\ {}^{\alpha}g_{ij}] \\
&\quad -\ {}^{\alpha}g_{cd}\ \ {}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}(\ {}^{\alpha}N_i^c\ {}^{\alpha}N_j^d), \qquad (30) \\
{}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}g_{ab} &= -2\ {}_{\shortmid}^{\alpha}R_{ab} + 2\ {}^{\alpha}\lambda\ {}^{\alpha}g_{ab}, \qquad (31) \\
{}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}(\ {}^{\alpha}N_j^e\ {}^{\alpha}g_{ae}) &= -2\ {}_{\shortmid}^{\alpha}R_{ia} + 2\ {}^{\alpha}\lambda\ {}^{\alpha}N_j^e\ \ {}^{\alpha}g_{ae}, \qquad (32)
\end{aligned}
$$

where ${}^{\alpha}\lambda =\ {}^{\alpha}r/5$, with ${}^{\alpha}r = \int\ {}_{\shortmid}^{\alpha}R\ {}^{\alpha}d\ {}^{\alpha}V/\ {}^{\alpha}d\ {}^{\alpha}V$, and the metric coefficients are those for (9) parametrized by ansatz (11), with respect to a fractional local coordinate basis (4).

A fractional differential geometry is modelled by nonholonomic integro–differential structures. A self–consistent system of fractional equations has to be N–adapted. We change in (30)–(32) the corresponding values: ${}^{\alpha}\nabla \rightarrow {}^{\alpha}\widehat{\mathbf{D}}$ and ${}_{\shortmid}^{\alpha}R_{\alpha\beta} \rightarrow\ {}^{\alpha}\widehat{\mathbf{R}}_{\alpha\beta}$. The resulting N–adapted fractional evolution equations for Ricci flows of symmetric fractional metrics, with respect to local coordinate frames (4), are

$$
\begin{aligned}
{}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}g_{ij} &= 2[\ {}^{\alpha}N_i^a\ {}^{\alpha}N_j^b\ (\ {}^{\alpha}\underline{\widehat{R}}_{ab} -\ {}^{\alpha}\lambda\ {}^{\alpha}g_{ab}) -\ {}^{\alpha}\underline{\widehat{R}}_{ij} +\ {}^{\alpha}\lambda\ {}^{\alpha}g_{ij}] \\
&\quad -\ {}^{\alpha}g_{cd}\ \ {}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}(\ {}^{\alpha}N_i^c\ {}^{\alpha}N_j^d), \qquad (33) \\
{}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}g_{ab} &= -2\ \left(\ {}^{\alpha}\underline{\widehat{R}}_{ab} -\ {}^{\alpha}\lambda\ {}^{\alpha}g_{ab}\right), \qquad (34) \\
{}^{\alpha}\widehat{R}_{ia} &= 0 \text{ and }\ {}^{\alpha}\widehat{R}_{ai} = 0, \qquad (35)
\end{aligned}
$$

where the fractional Ricci tensor coefficients ${}^{\alpha}\underline{\widehat{R}}_{ij}$ and ${}^{\alpha}\underline{\widehat{R}}_{ab}$ are computed with respect to coordinate coframes (4), being frame transforms (12) of the corresponding formulas (18) defined with respect to N–adapted coframes (8). The equations (35) constrain the nonholonomic fractional Ricci flows to result in symmetric fractional metrics. In general, fractional geometries are with nonholonomic integro–differential structures resulting in nonsymmetric fractional metrics, a similar conclusion for integer dimensions was proven in Ref. [10].

4.1 Main Theorems on fractional Ricci flows

One of the most important Perelman's contributions to the theory of Ricci flows was that he proved that Hamilton's evolution equations, in some

adapted forms, can be derived from certain functionals following a variational procedure. We show that Perelman's approach can be generalized to a fractional N–adapted formalism for evolution of geometric objects. In explicit form, we show how equations of type (33) and (34) can be derived by a fractional integro–differential calculus (for simplicity, we take a zero normalized term with ${}^{\alpha}\lambda = 0$).

Definition 4.1 *A general fractional metric ${}^{\alpha}\mathbf{g}$ evolving via a general fractional Ricci flow is called a breather if for some $\chi_1 < \chi_2$ and $\beta > 0$ the metrics $\beta\ {}^{\alpha}\mathbf{g}(\chi_1)$ and $\beta\ {}^{\alpha}\mathbf{g}(\chi_2)$ differ only by a fractional diffeomorphism preserving the Whitney sum (5). The cases $\beta =, <, > 1$ define correspondingly the steady, shrinking and expanding breathers.*

We note that because of nonholonomic character of fractional evolution we can model processes when, for instance, the h–component of metric is steady but the v–component is shrinking. Clearly, the expending properties depend on the type of calculus and connections are used for definition of Ricci flows.

Following a N–adapted variational calculus for ${}^{\alpha}\widehat{\mathcal{F}}({}^{\alpha}\mathbf{g}, \overset{\alpha}{\mathbf{N}}, {}^{\alpha}\widehat{f})$, see Lemma 3.1, with Laplacian ${}^{\alpha}\widehat{\Delta}$ and h- and v–components of the Ricci tensor, ${}^{\alpha}\widehat{R}_{ij}$ and ${}^{\alpha}\widehat{S}_{ij}$, defined by ${}^{\alpha}\widehat{\mathbf{D}}$ and considering parameter $\tau(\chi)$, $\partial\tau/\partial\chi = -1$, we prove

Theorem 4.1 *The fractional N–adapted Ricci flows are characterized by evolution equations*

$$\begin{aligned} {}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}\underline{g}_{ij} &= -2\ {}^{\alpha}\underline{\widehat{R}}_{ij},\quad {}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ \underline{g}_{ab}\partial\chi = -2\ {}^{\alpha}\underline{\widehat{R}}_{ab}, \\ {}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}\widehat{f} &= -\ {}^{\alpha}\widehat{\Delta}\ {}^{\alpha}\widehat{f} + \left|\ {}^{\alpha}\widehat{\mathbf{D}}\ {}^{\alpha}\widehat{f}\right|^2 -\ {}^{\alpha}R -\ {}^{\alpha}S \end{aligned}$$

and the properties that $\int\limits_{{}^{\alpha}\mathbf{V}} e^{-\ {}^{\alpha}\widehat{f}}\ {}^{\alpha}d\ {}^{\alpha}V = const$ *and*

$$\begin{aligned} &{}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}\widehat{\mathcal{F}}({}^{\alpha}\mathbf{g}(\chi), \overset{\alpha}{\mathbf{N}}(\chi),\ {}^{\alpha}\widehat{f}(\chi)) = 2\int\limits_{{}^{\alpha}\mathbf{V}} [|\ {}^{\alpha}\widehat{R}_{ij} +\ {}^{\alpha}D_i\ {}^{\alpha}D_j\ {}^{\alpha}\widehat{f}|^2 \\ &+|\ {}^{\alpha}\widehat{R}_{ab} +\ {}^{\alpha}D_a\ {}^{\alpha}D_b\ {}^{\alpha}\widehat{f}|^2]e^{-\ {}^{\alpha}\widehat{f}}\ {}^{\alpha}d\ {}^{\alpha}V. \end{aligned}$$

Proof. Such a proof which is very similar to those for Riemannian spaces, originally proposed by G. Perelman [2], see also details in the Proposition 1.5.3 of [5], and nonholonomic manifolds (additional remarks on the

canonical d–connection $\widehat{\mathbf{D}}$ on nonholonomic manifolds of integer dimension given in [8]). All those constructions can be reproduced in N–adapted fractional form using ${}^{\alpha}\widehat{\mathbf{D}}$ with coefficients (21).□

A similar analogy of calculus with ${}^{\alpha}\widehat{\mathbf{D}}$ to that for the integer case with ∇ allows us to generalize the formulation and proof of Proposition 1.5.8 in [5] containing the details of the original result from [2]), resulting in:

Theorem 4.2 *If a family of fractional metric* ${}^{\alpha}\mathbf{g}(\chi)$, *fractional function* ${}^{\alpha}\widehat{f}(\chi)$ *and parameter function* $\widehat{\tau}(\chi)$ *evolve subjected to the conditions of the system of equations*

$$
\begin{aligned}
{}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}\underline{g}_{ij} &= -2\ {}^{\alpha}\underline{\widehat{R}}_{ij},\quad {}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}\underline{g}_{ab} = -2\ {}^{\alpha}\underline{\widehat{R}}_{ab},\\
{}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}\widehat{f} &= -\ {}^{\alpha}\widehat{\Delta}\ {}^{\alpha}\widehat{f} + \left|\ {}^{\alpha}\widehat{\mathbf{D}}\ {}^{\alpha}\widehat{f}\right|^{2} - {}^{\alpha}R - {}^{\alpha}S + \frac{n+m}{2\widehat{\tau}},\\
{}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\widehat{\tau} &= -1,
\end{aligned}
$$

there are satisfied the properties $\int_{{}^{\alpha}\mathbf{V}} (4\pi\widehat{\tau})^{-(n+m)/2} e^{-\ {}^{\alpha}\widehat{f}}\ {}^{\alpha}d\ {}^{\alpha}V = const$ *and*

$$
\begin{aligned}
&{}_{1\chi}\overset{\alpha}{\underline{\partial}}_{\chi}\ {}^{\alpha}\widehat{\mathcal{W}}(\ {}^{\alpha}\mathbf{g}(\chi), \overset{\alpha}{\mathbf{N}}(\chi),\ {}^{\alpha}\widehat{f}(\chi), \widehat{\tau}(\chi)) = \\
&2\int_{\mathbf{V}} \widehat{\tau}[|\ {}^{\alpha}\widehat{R}_{ij} + {}^{\alpha}D_i\ {}^{\alpha}D_j\ {}^{\alpha}\widehat{f} - \frac{1}{2\widehat{\tau}}\ {}^{\alpha}g_{ij}|^2 + \\
&|\ {}^{\alpha}\widehat{R}_{ab} + {}^{\alpha}D_a\ {}^{\alpha}D_b\ {}^{\alpha}\widehat{f} - \frac{1}{2\widehat{\tau}}\ {}^{\alpha}g_{ab}|^2](4\pi\widehat{\tau})^{-(n+m)/2} e^{-\ {}^{\alpha}\widehat{f}}\ {}^{\alpha}d\ {}^{\alpha}V.
\end{aligned}
$$

The functional ${}^{\alpha}\mathcal{W}(\ {}^{\alpha}\mathbf{g}(\chi), \overset{\alpha}{\mathbf{N}}(\chi),\ {}^{\alpha}f(\chi), \tau(\chi))$ is nondecreasing in time and the monotonicity is strict unless we are on a shrinking fractional gradient soliton. This property depends on the type of fractional d–connection, or covariant connection we use.

Corollary 4.1 *The fractional evolution, for all time* $\tau \in [0, \tau_0)$, *of N–adapted frames*

$$
{}^{\alpha}\mathbf{e}_{\alpha}(\tau) = {}^{\alpha}\mathbf{e}_{\alpha}^{\ \underline{\alpha}}(\tau, u)\ {}^{\alpha}\partial_{\underline{\alpha}}
$$

is defined by the coefficients

$$
{}^{\alpha}\mathbf{e}_{\alpha}^{\ \underline{\alpha}}(\tau, u) = \begin{bmatrix} {}^{\alpha}e_{i}^{\ \underline{i}}(\tau, u) & {}^{\alpha}N_{i}^{b}(\tau, u)\ {}^{\alpha}e_{b}^{\ \underline{a}}(\tau, u) \\ 0 & {}^{\alpha}\ e_{a}^{\ \underline{a}}(\tau, u) \end{bmatrix},
$$

with

$$
{}^{\alpha}g_{ij}(\tau) = {}^{\alpha}e_{i}^{\ \underline{i}}(\tau, u)\ {}^{\alpha}e_{j}^{\ \underline{j}}(\tau, u)\eta_{\underline{ij}},
$$

where $\eta_{\underline{ij}} = diag[\pm 1, ... \pm 1]$ *establish the signature of* ${}^{\alpha}g^{[0]}_{\underline{\alpha\beta}}(u)$, *is given by equations*

$$ {}_{1}{}_{\tau}\overset{\alpha}{\underline{\partial}}_{\tau}\ {}^{\alpha}e^{\underline{\alpha}}_{\alpha} = {}^{\alpha}g^{\underline{\alpha\beta}}\ {}^{\alpha}_{\shortmid}R_{\underline{\beta\gamma}}\quad {}^{\alpha}e^{\underline{\gamma}}_{\alpha}, $$

if we prescribe fractional flows for the Levi–Civita connection ${}^{\alpha}\nabla$, *and*

$$ {}_{1}{}_{\tau}\overset{\alpha}{\underline{\partial}}_{\tau}\ {}^{\alpha}e^{\underline{\alpha}}_{\alpha} = {}^{\alpha}g^{\underline{\alpha\beta}}\ {}^{\alpha}\widehat{R}_{\underline{\beta\gamma}}\ {}^{\alpha}e^{\underline{\gamma}}_{\alpha}, $$

if we prescribe fractional flows for the canonical d–connection ${}^{\alpha}\widehat{\mathbf{D}}$.

We conclude that the fractional flows are characterized additionally by fractional evolutions of N–adapted frames (12) (see a similar proof for flows of integer dimension nonholonomic frames in [8]).

4.2 Statistical Analogy for Fractional Ricci Flows

A functional ${}^{\alpha}\widehat{\mathcal{W}}$ is in a sense analogous to minus entropy (such an interpretation was supposed by Grisha Perelman for Ricci flows of Riemannian metrics [2]). This allows us to elaborate a statistical model for fractional nonholonomic flows if the conditions of Theorem 4.2 are satisfied.[7]

For the partition function ${}^{\alpha}\widehat{Z} = \exp\{\int_{{}^{\alpha}\mathbf{V}}[-\ {}^{\alpha}\widehat{f} + \frac{n+m}{2}]\ {}^{\alpha}\widehat{\mu}\ {}^{\alpha}d\ {}^{\alpha}V\}$, we prove:

Theorem 4.3 *Any family of fractional nonholonomic geometries satisfying the fractional evolution equations for the canonical d–connection is charac-*

[7]Let us remember some concepts from statistical mechanics. The partition function $Z = \int \exp(-\beta E) d\omega(E)$ for the canonical ensemble at temperature β^{-1} is defined by the measure taken to be the density of states $\omega(E)$. The thermodynamical values are computed in the form: the average energy, $< E >= -\partial \log Z/\partial\beta$, the entropy $S = \beta < E > + \log Z$ and the fluctuation $\sigma =< (E- < E >)^2 >= \partial^2 \log Z/\partial\beta^2$.

terized by three thermodynamic values

$$
\begin{aligned}
\left\langle {}^{\alpha}\widehat{E} \right\rangle &= -\widehat{\tau}^{2} \int\limits_{{}^{\alpha}\mathbf{V}} \left({}^{\alpha}R + {}^{\alpha}S + \left| {}_{h}^{\alpha}D\ {}^{\alpha}\widehat{f} \right|^{2} + \left| {}_{v}^{\alpha}D\ {}^{\alpha}\widehat{f} \right|^{2} - \frac{n+m}{2\widehat{\tau}} \right) \\
&\quad \times {}^{\alpha}\widehat{\mu}\ {}^{\alpha}d\ {}^{\alpha}V, \\
{}^{\alpha}\widehat{S} &= -\int\limits_{{}^{\alpha}\mathbf{V}} [\widehat{\tau} \left({}^{\alpha}R + {}^{\alpha}S + \left| {}^{h}D\ {}^{\alpha}\widehat{f} \right|^{2} + \left| {}^{v}D\ {}^{\alpha}\widehat{f} \right|^{2} \right) + {}^{\alpha}\widehat{f} \\
&\quad - \frac{n+m}{2}]\ {}^{\alpha}\widehat{\mu}\ \ {}^{\alpha}d\ {}^{\alpha}V, \\
{}^{\alpha}\widehat{\sigma} &= 2\ \widehat{\tau}^{4} \int\limits_{\mathbf{V}} [|\ {}^{\alpha}\widehat{R}_{ij} + {}^{\alpha}D_{i}\ {}^{\alpha}D_{j}\ {}^{\alpha}\widehat{f} - \frac{1}{2\widehat{\tau}}\ {}^{\alpha}g_{ij}|^{2} \\
&\quad +|\ {}^{\alpha}\widehat{R}_{ab} + {}^{\alpha}D_{a}\ {}^{\alpha}D_{b}\ {}^{\alpha}\widehat{f} - \frac{1}{2\widehat{\tau}}\ {}^{\alpha}g_{ab}|^{2}]\ {}^{\alpha}\widehat{\mu}\ \ {}^{\alpha}d\ {}^{\alpha}V.
\end{aligned}
$$

A fractional, or integer, differential geometry defined by corresponding fundamental geometric objects and a fixed, in general, non–integer differential system is thermodynamically more convenient in dependence of the values of the above mentioned characteristics of Ricci flow evolution.

Conclusion 4.1 *Finally, we draw the conclusions:*

- *There is a version of fractional differential and integral calculus based on the left Caputo derivative when the resulting models of fractional differential geometry are with a N–connection adapted calculus similarly to noholonomic manifolds and Finsler–Lagrange geometry.*

- *A Ricci flow theory of fractional geometries can be considered as a nonholonomic evolution model transforming standard integer metrics and connections (for instance, in Riemann geometry) into generalized ones on nonsymmetric/noncommutative/fractional ... spaces.*

- *A very important property of fractional calculus theories and related geometric and physical models is that we can work with more "singular" functions and field interaction/evolution models in physics and applied mathematics.*

A Fractional Integro–Differential Calculus on $\mathbb{R}^n$

We summarize the formalism for a vector fractional differential and integral calculus elaborated on "flat" spaces [24]. The constructions involve

a fundamental theorem of calculus and fractional integral Green's, Stokes' and Gauss's theorems, which are important for definition, in this work, of fractional Perelman's functionals.

A.1 Riemann–Liouville and Caputo fractional derivatives

It is possible to elaborate different types of models of fractional geometry using different types of fractional derivatives. We follow an approach when the geometric constructions are most closed to "integer" calculus.

A.1.1 Left and right fractional RL derivatives

Let us consider that $f(x)$ is a derivable function $f : [\ _1x,\ _2x] \to \mathbb{R}$, for $\mathbb{R} \ni \alpha > 0$, and denote the derivative on x as $\partial_x = \partial/\partial x$.

The left Riemann–Liouville (RL) derivative is

$$ {}_{1x}\overset{\alpha}{\partial}_x f(x) := \frac{1}{\Gamma(s-\alpha)} \left(\frac{\partial}{\partial x}\right)^s \int\limits_{_1x}^{x} (x-\ x')^{s-\alpha-1} f(x') dx', $$

where Γ is the Euler's gamma function. The left fractional Liouville derivative of order α, where $s-1<\alpha<s$, with respect to coordinate x is defined $\overset{\alpha}{\partial}_x f(x) := \lim_{_1x\to-\infty}\ {}_{1x}\overset{\alpha}{\partial}_x f(x)$.

The right RL derivative is ${}_x\overset{\alpha}{\partial}\ _{2x} f(x) := \frac{1}{\Gamma(s-\alpha)} \left(-\frac{\partial}{\partial x}\right)^s \int\limits_{x}^{_2x} (x'-x)^{s-\alpha-1}$ $f(x')dx'$. The corresponding right fractional Liouville derivative is ${}_x\overset{\alpha}{\partial} f(x^k) :=$ $\lim_{_2x\to\infty}\ {}_x\overset{\alpha}{\partial}\ _{2x} f(x)$. In this work, we shall not use right derivatives.

Only the fractional Liouville derivatives define operators satisfying the semigroup properties on function spaces. The fractional RL derivative of a constant C is not zero but, for instance, ${}_{1x}\overset{\alpha}{\partial}_x C = C\frac{(x-\ _1x)^{-\alpha}}{\Gamma(1-\alpha)}$. Complete fractional integro–differential constructions based only on such derivatives seem to be very cumbersome and has a number of properties which are very different from similar ones for integer calculus.

A.1.2 Fractional Caputo derivatives

The respective left and right fractional Caputo derivatives are

$$
\begin{aligned}
{}_{1x}\overset{\alpha}{\underline{\partial}}_{x} f(x) &:= \frac{1}{\Gamma(s-\alpha)} \int\limits_{1x}^{x} (x - x')^{s-\alpha-1} \left(\frac{\partial}{\partial x'}\right)^{s} f(x') dx', \quad \text{(A.1)} \\
\text{and} \quad {}_{x}\overset{\alpha}{\underline{\partial}}_{2x} f(x) &:= \frac{1}{\Gamma(s-\alpha)} \int\limits_{x}^{2x} (x' - x)^{s-\alpha-1} \left(-\frac{\partial}{\partial x'}\right)^{s} f(x') dx' ,
\end{aligned}
$$

where we underline the partial derivative symbol, $\underline{\partial}$, in order to distinguish the Caputo operators from the RL ones with usual ∂. A very important property is that for a constant C, for instance, ${}_{1x}\overset{\alpha}{\underline{\partial}}_{x} C = 0$. In our approach, we shall give priority to the fractional left Caputo derivative resulting in constructions which are very similar to those with integer calculus.

A.2 Vector operations and non–integer differential forms

A.2.1 Fractional integral

To formulate fractional integral (Gauss's, Stokes', Green's etc) theorems with a formal noninteger order α integral ${}_{1x}\overset{\alpha}{I}{}_{2x}$, we need a generalization of the Newton–Leibniz formula ${}_{1x}\overset{\alpha}{I}{}_{2x}\left({}_{1x}\overset{\alpha}{\partial}_{x} f(x)\right) = f({}_{2}x) - f({}_{1}x)$, for a fractional derivative ${}_{1x}\overset{\alpha}{\partial}_{x}$. Such "mutually inverse" operations do not exist for an arbitrary taken type of fractional derivative.

Let us denote by $L_z({}_{1}x, {}_{2}x)$ the set of those Lebesgue measurable functions f on $[{}_{1}x, {}_{2}x]$ for which $||f||_z = (\int\limits_{1x}^{2x} |f(x)|^z dx)^{1/z} < \infty$. We write $C^z[{}_{1}x, {}_{2}x]$ is a space of functions, which are z times continuously differentiable on this interval.

Using the fundamental theorem of fractional calculus [24], we have that for any real–valued function $f(x)$ defined on a closed interval $[{}_{1}x, {}_{2}x]$, there is a function $F(x) = {}_{1x}\overset{\alpha}{I}_{x} f(x)$ defined by the fractional Riemann–Liouville integral ${}_{1x}\overset{\alpha}{I}_{x} f(x) := \frac{1}{\Gamma(\alpha)} \int\limits_{1x}^{x} (x - x')^{\alpha-1} f(x') dx'$, when the function

$f(x) = {}_{1x}\overset{\alpha}{\underline{\partial}}_x F(x)$ satisfies the conditions

$$\begin{aligned} {}_{1x}\overset{\alpha}{\underline{\partial}}_x \left({}_{1x}\overset{\alpha}{I}_x f(x) \right) &= f(x),\ \alpha > 0, \\ {}_{1x}\overset{\alpha}{I}_x \left({}_{1x}\overset{\alpha}{\underline{\partial}}_x F(x) \right) &= F(x) - F({}_1x),\ 0 < \alpha < 1, \end{aligned}$$

for all $x \in [{}_1x,\ {}_2x]$. So, the right fractional RL integral is inverse to the right fractional Caputo derivative.

There is a corresponding fractional generalization for the Taylor formula

$$f({}_2x) - f({}_1x) = {}_{1x}\overset{\alpha}{I}_x \left({}_{1x}\overset{\alpha}{\underline{\partial}}_x f(x) \right) + \sum_{s_1=0}^{s-1} \frac{1}{s_1!} ({}_2x - {}_1x)^{s_1} f^{(s_1)}({}_1x),$$

for $s - 1 < \alpha \leq s$, where $f^{(s_1)}(x) = {}_{1x}\overset{\alpha}{\underline{\partial}}_x f(x)$.

A.2.2 Definition of fractional vector operations

Let X be a domain of $\mathbb{R}^n$ parametrized as $X = \{{}_1x^i \leq x^i \leq {}_2x^i\}$ (or, in brief, $X = \{{}_1x \leq x \leq {}_2x\}$, which substitute the closed one dimensional interval $[{}_1x,\ {}_2x]$. Let $f(x^i)$ and $F_k(x^i)$ be real–valued functions that have continuous derivatives up to order $k-1$ on X, such that the $k-1$ derivatives are absolutely continuous, i.e., $f, \mathbf{F} = \{F_i\} \in AC^k[X]$, see details in [19].

For a basis e^i on X, we can define a fractional generalization of gradient operator $\partial = e^i \partial_i = e^i \partial / \partial x^i$, when ${}_{1x}\overset{\alpha}{\underline{\partial}} = e^i\ {}_{1x}\overset{\alpha}{\underline{\partial}}_i$, for ${}_{1x}\overset{\alpha}{\underline{\partial}}_i := {}_{1x}\overset{\alpha}{\underline{\partial}}_{x^i}$ being the left fractional Caputo derivatives on x^i defined by (A.1). If $f(x) = f(x^i)$ is a $(k-1)$ times continuously differentiable scalar field such that $\partial_i^{k-1} f$ is absolutely continuous, we can define the fractional gradient of f,$\overset{\alpha}{grad}\ f := {}_{1x}\overset{\alpha}{\partial} f = e^i\ {}_{1x}\overset{\alpha}{\underline{\partial}}_i f$. Let us consider that $X \subset \mathbb{R}^n$ has a flat metric η_{ij} and its inverse η^{ij}. Then we can define $F^i = \eta^{ij} F_j(x^k)$ and construct the fractional divergence operator$\overset{\alpha}{div}\mathbf{F} : = {}_{1x}\ \overset{\alpha}{\underline{\partial}}_i F^i(x^k)$.

We can not use the Leibniz rule in a fractional generalization of the vector calculus because for two analytic functions ${}^1f$ and ${}^2f$ we have

$$\begin{aligned} \overset{\alpha}{grad}({}^1f\ {}^2f) &\neq (\overset{\alpha}{grad}\ {}^1f)\ {}^2f + (\overset{\alpha}{grad}\ {}^2f)\ {}^1f, \\ \overset{\alpha}{div}(f\mathbf{F}) &\neq (\overset{\alpha}{grad}\ f\ , \mathbf{F}) + f\ \overset{\alpha}{div}\mathbf{F}. \end{aligned}$$

This follows from the property that ${}_{1x'}\overset{\alpha}{\underline{\partial}}_{i'}({}^1f\ (x^{j'})\ {}^2f(x^{k'})) \neq$ $({}_{1x'}\overset{\alpha}{\underline{\partial}}_{i'}\ {}^1f(x^{j'}))\ {}^2f(x^{k'}) + ({}_{1x'}\overset{\alpha}{\underline{\partial}}_{i'}\ {}^2f(x^{k'}))\ {}^1f(x^{j'})$.

A fractional volume integral is a triple fractional integral within a region $X \subset \mathbb{R}^3$, for instance, of a scalar field $f(x^k)$, $\overset{\alpha}{I}(f) = \overset{\alpha}{I}[x^k]f(x^k) = \overset{\alpha}{I}[x^1]\ \overset{\alpha}{I}[x^2]\ \overset{\alpha}{I}[x^3]f(x^k)$. For $\alpha = 1$ and $f(x,y,z)$, we have $\overset{\alpha}{I}(f) = \iiint\limits_X dVf = \iiint\limits_X dxdydz\ f$.

The fundamental fractional integral theorems with $\overset{\alpha}{grad}$, $\overset{\alpha}{div}$ etc are considered in [24] for the fractional Caputo derivatives. We omit such details in this work, but emphasize that a self–consistent fractional integro–differential calculus on fractional manifolds, which is very similar to the "integer" one, can be developed following above constructions.

A.2.3 Fractional differential forms

There were elaborated different approaches to fractional differential form caluculus. For instance, a fractional generalization of differential has been presented by Ben Adda, see review of his results in [27]. Then different fractional generalizations of differential forms and application to dynamical systems were proposed, see critical remarks and original results in [24, 25] (fractional differentials/forms were constructed using different fractional derivatives etc but fractional integral theorems being considered only recently in [24]).

Following [24], an exterior fractional differential is defined through the fractional Caputo derivatives which is self–consistent with the definition of fractional integral considered above. We introduce for ${}_1x^i = 0$ the fractional absolute differential $\overset{\alpha}{d}$ in the form

$$\overset{\alpha}{d} := (dx^j)^\alpha \quad {}_0\overset{\alpha}{\underline{\partial}}_j, \text{ where } \overset{\alpha}{d}x^j = (dx^j)^\alpha \frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)}.$$

For the "integer" calculus, we use as local coordinate co-bases/–frames the differentials $dx^j = (dx^j)^{\alpha=1}$. The "fractional" symbol $(dx^j)^\alpha$, related to $\overset{\alpha}{d}x^j$, is used instead of dx^i for elaborating a co–vector/differential form calculus, see below the formula (A.5). It is considered that for $0 < \alpha < 1$ we have $dx = (dx)^{1-\alpha}(dx)^\alpha$.

An exterior fractional differential can be defined through the fractional Caputo derivatives in the form

$$\overset{\alpha}{d} = \sum_{j=1}^{n} \Gamma(2-\alpha)(x^j)^{\alpha-1}\ \overset{\alpha}{d}x^j \quad {}_0\overset{\alpha}{\underline{\partial}}_j.$$

Differentials are dual to partial derivatives, and derivation is inverse to integration. For a fractional calculus, the concept of "dual" and "inverse" have a more sophisticate relation to "integration" and, in result, there is a more complex relation between forms and vectors.

The fractional integration for differential forms on $L = [\ _1x,\ _2x]$ is introduced using the operator $\ _L\overset{\alpha}{I}[x] := \int\limits_{_1x}^{_2x} \frac{(dx)^{1-\alpha}}{\Gamma(\alpha)(\ _2x - x)^{1-\alpha}}$ when, for $0 < \alpha < 1$,

$$\ _L\overset{\alpha}{I}[x] \quad _{_1x}\overset{\alpha}{d}_x f(x) = f(\ _2x) - f(\ _1x). \tag{A.2}$$

The **fractional differential** of a function $f(x)$ is $\ _{_1x}\overset{\alpha}{d}_x f(x) = [...]$, with square brackets considered are defined by formula

$$\int\limits_{_1x}^{_2x} \frac{(dx)^{1-\alpha}}{\Gamma(\alpha)(\ _2x - x)^{1-\alpha}} \left[(dx')^{\alpha} \quad _{_1x}\overset{\alpha}{\underline{\partial}}_{x''} f(x'') \right] = f(x) - f(\ _1x).$$

The exact fractional differential 0–form is a fractional differential of the function

$$\ _{_1x}\overset{\alpha}{d}_x f(x) := (dx)^{\alpha} \quad _{_1x}\overset{\alpha}{\underline{\partial}}_{x'} f(x')$$

when the equation (A.2) is considered as the fractional generalization of the integral for a differential 1–form. So, the **fractional exterior derivative** can be written

$$\ _{_1x}\overset{\alpha}{d}_x := (dx^i)^{\alpha} \quad _{_1x}\overset{\alpha}{\underline{\partial}}_i. \tag{A.3}$$

Then, the fractional differential 1–form $\overset{\alpha}{\omega}$ with coefficients $\{F_i(x^k)\}$ is

$$\overset{\alpha}{\omega} = (dx^i)^{\alpha}\ F_i(x^k) \tag{A.4}$$

The exterior fractional derivatives of 1–form $\overset{\alpha}{\omega}$ gives a fractional 2–form,

$$\ _{_1x}\overset{\alpha}{d}_x(\ \overset{\alpha}{\omega}) = (dx^i)^{\alpha} \wedge (dx^j)^{\alpha} \quad _{_1x}\overset{\alpha}{\underline{\partial}}_j\ F_i(x^k).$$

This rule can be proven following the relations [19] that for any type fractional derivative $\overset{\alpha}{\partial}_x$, we have

$$\overset{\alpha}{\partial}_x\ (\ ^1f\ \ ^2f) = \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array} \right) \left(\overset{\alpha-k}{\partial}_x\ \ ^1f \right) \ \overset{\alpha=k}{\partial}_x\ (\ ^2f)\,,$$

for integer k, where

$$\left(\begin{array}{c} \alpha \\ k \end{array} \right) = \frac{(-1)^{k-1}\alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)} \text{ and } \quad \overset{k}{\partial}_x\ (dx)^{\alpha} = 0, k \geq 1.$$

There are used also the properties: ${}_{1x'}\overset{\alpha}{\partial}_{x'}\ (x'-{}_{1}x')^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-{}_{1}x)^{\beta-\alpha}$, where $n-1<\alpha<n$ and $\beta>n$, and ${}_{1x'}\overset{\alpha}{\partial}_{x'}\ (x'-{}_{1}x')^{k}=0$ for $k=0,1,2,...,n-1$. We obtain ${}_{1x}\overset{\alpha}{d}_{x}(x-{}_{1}x)^{\alpha}=(dx)^{\alpha}\ {}_{1x}\overset{\alpha}{\underline{\partial}}_{i'}x^{i'}=(dx)^{\alpha}\Gamma(\alpha+1)$, i.e.

$$(dx)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\ {}_{1x}\overset{\alpha}{d}_{x}(x-{}_{1}x)^{\alpha} \tag{A.5}$$

and write the fractional exterior derivative (A.3) in the form

$${}_{1x}\overset{\alpha}{d}_{x}:=\frac{1}{\Gamma(\alpha+1)}\ {}_{1x}\overset{\alpha}{d}_{x}(x^{i}-{}_{1}x^{i})^{\alpha}\ {}_{1x}\overset{\alpha}{\underline{\partial}}_{i}.$$

Using this formula, the fractional differential 1–form (A.4) can be written alternatively

$$\overset{\alpha}{\omega}=\frac{1}{\Gamma(\alpha+1)}\ {}_{1x}\overset{\alpha}{d}_{x}(x^{i}-{}_{1}x^{i})^{\alpha}\ F_{i}(x).$$

Having a well defined exterior calculus of fractional differential forms on flat spaces $\mathbb{R}^{n}$, we can generalize the constructions for a real manifold $M, \dim M=n$.

References

[1] R. S. Hamilton, Three–manifolds with positive Ricci curvature, J. Diff. Geom. **17** (1982) 255–306

[2] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv: math.DG/ 0211159

[3] G. Perelman, Ricci flow with surgery on three–manifolds, arXiv: math. DG/ 03109

[4] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three–manifolds, arXiv: math.DG/ 0307245

[5] H. -D. Cao and X. -P. Zhu, Hamilton–Perelman's proof of the Poincar'e conjecture and the geometrization conjecture, Asian J. Math., **10** (2006) 165–495; arXiv: math. DG/ 0612069

[6] B. Kleiner and J. Lott, Notes on Perelman's papers, arXiv: math.DG/ 0605667

[7] J. W. Morgan and G. Tian, Ricci flow and the Poincaré conjecture, arXiv: math.DG/ 0607607

[8] S. Vacaru, Nonholonomic Ricci flows: II. Evolution equations and dynamics, J. Math. Phys. **49** (2008) 043504; arXiv: math.DG/ 0702598

[9] S. Vacaru, The Entropy of Lagrange–Finsler spaces and Ricci flows, Rep. Math. Phys. **63** (2009) 95-110; arXiv: math.DG/ 0701621

[10] S. Vacaru, Nonholonomic Ricci flows, exact solutions in gravity, and symmetric and nonsymmetric metrics, Int. J. Theor. Phys. **48** (2009) 579-606; arXiv: 0806.3812

[11] S. Vacaru, Parametric nonholonomic frame transforms and exact solutions in gravity, Int. J. Geom. Methods. Mod. Phys. (IJGMMP) **4** (2007) 1285-1334; arXiv: 0704.3986

[12] S. Vacaru, Finsler and Lagrange geometries in Einstein and string gravity, Int. J. Geom. Methods. Mod. Phys. (IJGMMP) **5** (2008) 473-511; arXiv: 0907.4278

[13] Clifford and Riemann- Finsler Structures in Geometric Mechanics and Gravity, Selected Works, by S. Vacaru, P. Stavrinos, E. Gaburov and D. Gonţa. Differential Geometry – Dynamical Systems, Monograph 7 (Geometry Balkan Press, 2006); www.mathem.pub.ro/dgds/mono/va-t.pdf and arXiv: gr-qc/0508023

[14] R. Miron and M. Anastasiei, The Geometry of Lagrange Spaces: Theory and Applications, FTPH no. **59** (Kluwer Academic Publishers, Dordrecht, Boston, London, 1994)

[15] A. Bejancu and H. R. Farran, Foliations and Geometric Structures (Springer, 2005)

[16] B. Ross, A brief history and exposition of the fundamental theory of fractional calculus, Lect. Notes Math. **457** (1975) 1–36

[17] K. B. Oldham and J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order (Academic Press, NY, 1974)

[18] I. Podlubny, Fractional Differential Equatinons (Academic Press, San Diego, 1999)

[19] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Application of Fractional Differential Equations (Elsevier, Amsterdam, 2006)

[20] K. Nishimoto, Fractional Calculus: Integrations and Differentiations of Arbitrary Order (University of New Haven Press, New Haven, 1989)

[21] R. Hilfer (Ed.), Applications ofr Fractional Calculus in Physics (World Scientific, Singapore, 2000)

[22] B. West, M. Bologna and P. Grigolini, Physics of Fractal Operators (Springer, NY, 2003)

[23] G. M. Zaslavsky, Chaos, fractional kinetics and anomalous transport, Phys. Rep. **371** (2002) 461–580

[24] V. E. Tarasov, Fractional vector calculus and fractional Maxwell's equations, Annals of Phys. (NY) **323** (2008) 2756–2778; arXiv: 0907.2363

[25] V. E. Tarasov, Fractional generalization of gradient systems. Lett. Math. Phys. **73** (2005) 49–58

[26] D. Baleanu and S. Muslih, On fractional variational principles, Adv. Frac. Calc. Part 2 (2007) 115–126

[27] F. Ben Adda, The differentiability in fractional calculus, Nonlinear Analysis **47** (2001) 5423–5428

[28] I. D. Albu, M. Neamţu and D. Opriş, The geometry of fractional osculator bundle of higher order and applications, arXiv: 0709.2000v1

[29] S. Vacaru, Superstrings in Higher Order Extensions of Finsler Superspaces, Nucl. Phys. B, **434** (1997) 590 -656; arXiv: hep-th/ 9611034

[30] S. Vacaru, Exact Solutions with Noncommutative Symmetries in Einstein and Gauge Gravity, J. Math. Phys. **46** (2005) 042503; arXiv: gr-qc/ 0307103

[31] S. Vacaru, Branes and quantization for an A–model complexification of Einstein gravity in Almost Kähler variables, Int. J. Geom. Methods. Mod. Phys. (IJGMMP) **6** (2009) 873-909; arXiv: 0810.4692

[32] J. Munkhammar, Riemann–Liouville fractional Einstein field equations, arXiv: 1003.4981

[33] S. Vacaru, On general solutions in Einstein and higher dimension gravity, Int. J. Theor. Phys. **49** (2010) 884-913; arXiv: 0909.3949v4

[34] S. Vacaru, Nonholonomic Ricci flows: III. Curve flows and solitonic hierarchies, arXiv: 0704.2062

[35] S. Vacaru, Nonholonomic Ricci flows: Exact solutions and gravity, Electronic Journal of Theoretical Physics (EJTP) **6**, N20, (2009) 27—58; arXiv: 0705.0728

[36] S. Vacaru, Nonholonomic Ricci flows and parametric deformations of the solitonic pp-Waves and Schwarzschild solutions, Electronic Journal of Theoretical Physics (EJTP) **6**, N21 (2009) 63-93; arXiv: 0705.0729

[37] S. Vacaru, Ricci flows and solitonic pp–waves, Int. J. Mod. Phys. A **21** (2006) 4899-4912, arXiv: hep-th/ 0602063

[38] S. Vacaru and M. Visinescu, Nonholonomic Ricci flows and running cosmological constant: I. 4D Taub-NUT metrics, Int. J. Mod. Phys. A **22** (2007) 1135-1159; arXiv: gr-qc/ 0609085